# Displaced vortices on lattices



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# Displaced vortices on lattices

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**Abstract.** The energy associated with a general displacement of either a single vortex, a cluster, or a superlattice of vortices on an arbitrary lattice is derived in terms of rapidly convergent sums of simple functions. Expansions to second order in the displacement, equivalent to the vibrational modes, are also derived for the single vortex, the superlattice, and an infinite row of vortices.

## 1. Introduction

Certain physical models, such as for Josephson-junction arrays, dense flux lines in high-$\kappa$ type-II superconductors, superfluids, the 2D Coulomb system, and screw dislocations in crystals under torsional stress are equivalent to a two-dimensional lattice of vortices characterised by a pairwise interaction energy proportional to $\log(r)$. For example, two-dimensional vortices in Eulerian fluid dynamics have a mutual interaction energy, $\mathbf{E}(r_{ij}) = -(\Gamma_i\Gamma_j d/4\pi)\log(|\boldsymbol{r}_i - \boldsymbol{r}_j|^2)$, where $\Gamma$ is the circulation and $d$ is the fluid density. Other systems have different constants, so the prototypical interaction energy is taken to be $-\log(|\boldsymbol{r}_i - \boldsymbol{r}_j|^2)$. Aside from these models, the vortex lattice is a non-trivial example of an infinite system whose constituents have long-ranged interactions. The emphasis here is on the change of energy associated with moving selected vortices from their lattice sites, with all other vortices remaining fixed or 'pinned'.

Even the simplest cases of triangular or square lattices of identical vortices produce a puzzle, at least superficially, in that their first-order stability with respect to single vortex displacements cannot be inferred from either of the two components of the system: the vortices and the uniform, static, neutralising background. Figure 1 shows the energy of a vortex interacting with six others located at the points of a regular hexagon. At the centre $\boldsymbol{r}_0$ of the hexagon the seventh vortex not only lacks first-order stability but has no energy variation of order $|\boldsymbol{r} - \boldsymbol{r}_0|^2$. (Of course, no collection of vortices can give an maximum or minimum in the energy, except at a vortex, because the resulting field is a solution of Poisson's equation.) Explicitly, the interaction energy in this case is

$$E(\boldsymbol{r}) = -\sum_{j=1}^{6} \log(|x + \mathrm{i}y - \mathrm{e}^{-\mathrm{i}\pi j/3}|^2)$$

$$= 2[x^6 - 15(x^4y^2 - x^2y^4) - y^6] + O((x, y)^{12}).$$

Because the entire infinite triangular lattice of vortices can be built by adding more hexagons with larger radii and appropriate orientation around the same origin it follows

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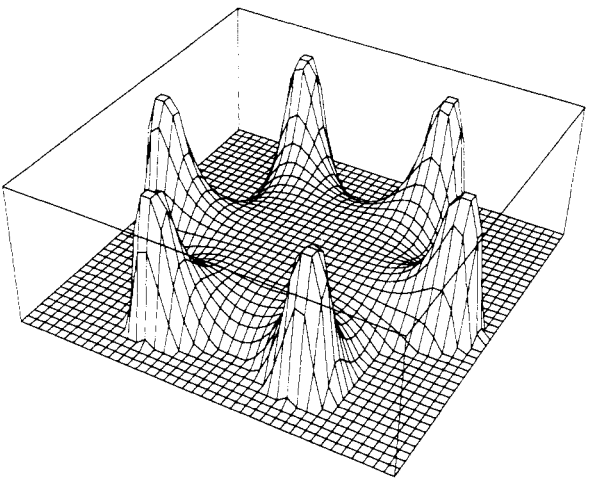

**Figure 1.** Energy $E(\boldsymbol{r})$ of a vortex interacting with six vortices on a regular hexagon. At the centre $\boldsymbol{r}_0$ of the hexagon the surface is flat through the fifth power of displacement $|\boldsymbol{r}-\boldsymbol{r}_0|$. (The energy range is bounded for clarity.)

that the infinite triangular lattice confers no first-order stability on single-vortex displacement in *any* direction. By symmetry, the infinite uniform background also would seem incapable of giving stability. The same result holds for the square lattice (displacement energy $\propto \delta r^4$). A closer look is required to derive the stability.

## 2. Notation

The energy density of an infinite lattice of vortices with strengths $\Gamma_j$ has the structure

$$E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) = \overline{\Gamma^2}\tilde{E}(\phi, \rho) + \frac{\overline{\Gamma^2}}{2}\log(N) + \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Gamma_i\Gamma_j f(\boldsymbol{r}_i - \boldsymbol{r}_j; \boldsymbol{L}_1, \boldsymbol{L}_2) \quad (1)$$

where $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ and $N$ is the number of vortices per unit cell defined by the generators $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ with the ratio $\rho = L_1/L_2$ and relative angle $\phi$: $\boldsymbol{L}_1 \cdot \boldsymbol{L}_2 = L_1 L_2 \cos\phi$. (Unit cells are those whose translations reproduce the lattice.) Explicit expressions are given in the appendix. Of the smallest unit cells the one having the largest $\phi_0 \leq \pi/2$ will be called elementary; it has generators $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ with the ratio $\rho_0 = e_1/e_2$ and relative angle $\phi_0$: $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = e_1 e_2 \cos\phi_0$. The set of lattice sites in a general or 'super lattice' unit cell will be denoted by $\mathbf{L} = \{\boldsymbol{r}_j : j = 1, \ldots, N\}$. This notation is illustrated in figure 2 for one vortex per elementary unit cell, the case that will be considered initially. For brevity, $f(\boldsymbol{r}_i - \boldsymbol{r}_j; \boldsymbol{L}_1, \boldsymbol{L}_2)$ will be denoted $f_L(\boldsymbol{r}_i - \boldsymbol{r}_j)$ or $f_L$ when there is no ambiguity.

## 3. Structure of the displacement energy

A simple, but useful, relation follows from the indifference of the energy density to the choice of superlattice unit cell. In particular,

$$E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) = E(\boldsymbol{r}_1; \boldsymbol{e}_1, \boldsymbol{e}_2) \quad (2)$$

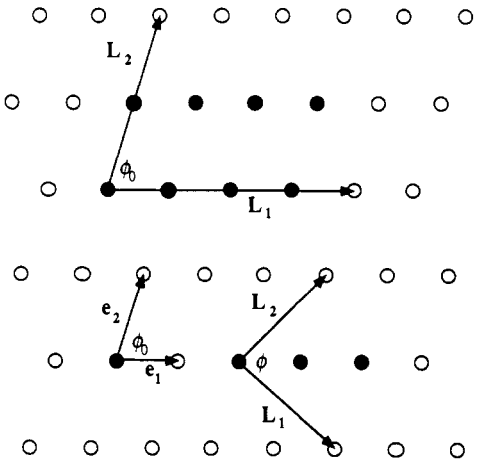


**Figure 2.** A simple vortex lattice, showing the elementary unit cell generators $e_1, e_2$ and two examples of superlattice unit cells having generators $L_1, L_2$. The superlattice unit cell at the top has generators colinear with $e_1, e_2$. The shaded vortices belong to the respective unit cells.

for the case of one vortex (with $\Gamma_1 = 1$) per unit cell. This immediately gives

$$\sum_{\mathsf{L}}^{[N]} f_L \equiv \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} f_L(\boldsymbol{r}_i - \boldsymbol{r}_j) = -\frac{N}{2}\log(N) - N(\tilde{E} - \tilde{E}_0) \tag{3}$$

where $\tilde{E}_0 \equiv \tilde{E}(\phi_0, \rho_0)$.

Now, consider the vortex-hole energy. Displacing a vortex from the lattice site $\boldsymbol{r}_1$ to $\boldsymbol{r}$ results in a superlattice of displacements corresponding to the generators $\boldsymbol{L}_1, \boldsymbol{L}_2$. The energy of a vortex-hole pair per unit cell (containing $N$ sites of the elementary lattice) is the energy difference

$$\begin{aligned}\mathsf{E}_{\mathrm{vh}}(\boldsymbol{r}, \boldsymbol{r}_1; \boldsymbol{L}_1, \boldsymbol{L}_2) &\equiv NE(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) - NE(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) \\ &= \sum_{\mathsf{L}}^{[N]} f_L(\boldsymbol{r} - \boldsymbol{r}_j) - \sum_{\mathsf{L}-\{\boldsymbol{r}_1\}}^{[N-1]} f_L(\boldsymbol{r}_1 - \boldsymbol{r}_j) - f_L(\boldsymbol{r} - \boldsymbol{r}_1) \\ &\equiv \mathsf{E}_{\mathrm{v}}(\boldsymbol{r}) + \mathsf{E}_{\mathrm{h}} - f_L(\boldsymbol{r} - \boldsymbol{r}_1)\end{aligned} \tag{4}$$

where the terms $\mathsf{E}_{\mathrm{v}}(\boldsymbol{r})$ (single-vortex energy) and $\mathsf{E}_{\mathrm{h}}$ (hole or vacancy energy) in the last line are defined by the corresponding terms in the previous line. The number of points that are summed is denoted above the summation signs and the set to which they belong is specified under the summation sign. The procedure in (4) is to cancel all terms except those involving $\boldsymbol{r}$ and $\boldsymbol{r}_1$. In general, the energy of $M \leqslant N$ vortex-hole pairs can be reduced to that of single vortex-hole pairs plus their mutual interactions

$$\begin{aligned}&\mathsf{E}_{\mathrm{vh}}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_M, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \\ &\quad = \sum_{j=1}^{M} \mathsf{E}_{\mathrm{vh}}(\boldsymbol{w}_j, \boldsymbol{r}_j) + \sum_{i<j}^{[M]} [f_L(\boldsymbol{w}_i - \boldsymbol{w}_j) + f_L(\boldsymbol{r}_i - \boldsymbol{r}_j) - f_L(\boldsymbol{w}_i - \boldsymbol{r}_j) - f_L(\boldsymbol{r}_i - \boldsymbol{w}_j)]\end{aligned} \tag{5}$$

where $\boldsymbol{w}_j$ are the positions of the displaced vortices and $\boldsymbol{r}_i$ the lattice sites of the holes.

To evaluate (4) consider first $\mathbf{E}_h$. The equivalence of all hole sites $\boldsymbol{r}_j$ in $\mathbf{L}$ allows $\mathbf{E}_h$ to be expressed as $1/N$ times the sum over all possible holes

$$\mathbf{E}_h = -\frac{1}{N}\sum_{j=1}^{N}\sum_{\mathbf{L}-\{\boldsymbol{r}_j\}}^{[N-1]} f_L(\boldsymbol{r}_j - \boldsymbol{r}_k) = -\frac{2}{N}\sum_{\mathbf{L}}^{[N]} f_L = \log(N) + 2(\tilde{E} - \tilde{E}_0) \tag{6}$$

where (3) was used to obtain the last equality.

Next, $\mathbf{E}_v(\boldsymbol{r})$ is periodic on the vortex lattice with positive singularities at $\boldsymbol{r} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2$ and is equal to $f_e(\boldsymbol{r}) \equiv f(\boldsymbol{r}; \boldsymbol{e}_1, \boldsymbol{e}_2)$. To see this, consider a particular lattice containing $2N$ vortices in the unit cell $\boldsymbol{L}_1, \boldsymbol{L}_2$. Of these, let $N$ be at the lattice sites $\mathbf{L}(\boldsymbol{r})$ of the simple lattice and denoted by $\boldsymbol{r}_j$. Let the remaining $N$, denoted by $\boldsymbol{w}_j$, be displaced from the simple lattice sites by a constant $\boldsymbol{r}$: $\boldsymbol{w}_j = \boldsymbol{r}_j + \boldsymbol{r}$; $j = 1, \ldots, N$. The energy per vortex of this lattice is, according to (1),

$$E(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) = \tilde{E} + \tfrac{1}{2}\log(2N) + \frac{1}{2N}\sum_{\mathbf{L}(\boldsymbol{r})+\mathbf{L}(\boldsymbol{w})}^{[2N]}\sum f_L. \tag{7}$$

The double sum above can be decomposed into sums over each lattice $\mathbf{L}(\boldsymbol{r})$ and $\mathbf{L}(\boldsymbol{w})$ separately, plus their cross term

$$\begin{aligned}\frac{1}{2N}\sum_{\mathbf{L}(\boldsymbol{r})+\mathbf{L}(\boldsymbol{w})}^{[2N]}\sum f_L &= \frac{1}{2N}2\sum_{\mathbf{L}(\boldsymbol{r})}^{[N]}\sum f_L + \frac{1}{2N}\sum_{\mathbf{L}(\boldsymbol{r})}^{N}\sum_{\mathbf{L}(\boldsymbol{w})}^{N} f_L(\boldsymbol{w}_i, \boldsymbol{r}_j) \\ &= -\tfrac{1}{2}\log(N) - (\tilde{E} - \tilde{E}_0) + \tfrac{1}{2}\mathbf{E}_v(\boldsymbol{r})\end{aligned} \tag{8}$$

where (3) and the definition (and periodicity) of $\mathbf{E}_v$ were used. However, the *same* energy per vortex given by (7) can equivalently be evaluated from (1) for the elementary unit cell $\boldsymbol{e}_1, \boldsymbol{e}_2$

$$E(\boldsymbol{w}_1, \boldsymbol{r}_1; \boldsymbol{e}_1, \boldsymbol{e}_2) = \tilde{E}_0 + \tfrac{1}{2}\log(2) + \tfrac{1}{2}f_e(\boldsymbol{r}). \tag{9}$$

Substituting (8) into (7) and equating with (9) gives the desired result:

$$\mathbf{E}_v(\boldsymbol{r}) \equiv \sum_{\mathbf{L}}^{[N]} f_L(\boldsymbol{r} - \boldsymbol{r}_j) = f_e(\boldsymbol{r}). \tag{10}$$

The last term on the right-hand side of (4), $f_L(\boldsymbol{r} - \boldsymbol{r}_1)$, is periodic on the unit cell $\boldsymbol{L}_1, \boldsymbol{L}_2$ with positive singularities at $\boldsymbol{r} = \boldsymbol{r}_1 + m_1\boldsymbol{L}_1 + m_2\boldsymbol{L}_2$. Moreover, it depends only on its argument normalised to the generators. In particular, for the case of super lattice unit cell generators collinear with the elementary unit cell generators, i.e. $\boldsymbol{L}_1 = n_1\boldsymbol{e}_1$, $\boldsymbol{L}_2 = n_2\boldsymbol{e}_2$, $n = 1, 2, 3, \ldots$, it follows that

$$f(q_1 n_1\boldsymbol{e}_1 + q_2 n_2\boldsymbol{e}_2; n_1\boldsymbol{e}_1, n_2\boldsymbol{e}_2) = f(q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2; \boldsymbol{e}_1, \boldsymbol{e}_2). \tag{11}$$

This collinearity is assumed hereafter, so it is sufficient to consider $f_e(\boldsymbol{r})$, which happens also to be $\mathbf{E}_v(\boldsymbol{r})$. The existence of only positive singularities for $f_e(\boldsymbol{r})$ raises the question of where the minima are found. For the triangular lattice ($\phi = 60°$, $\rho = 1$) the minima occur on the long diagonal of the elementary unit cell at $\boldsymbol{r}_{1/3} = c(\boldsymbol{e}_1 + \boldsymbol{e}_2)$, $c = \frac{1}{3}, \frac{2}{3}$. These locations are the centres of the elementary triangles of the triangular lattice. For the square lattice ($\phi = 90°$, $\rho = 1$) the minimum is at the centre of the elementary unit cell, $\boldsymbol{r}_{1/2} = \frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2)$. A contour plot is shown in figure 3 of $f_e(\boldsymbol{r})$ along the diagonal of an elementary unit cell as a function of $\phi$ for $\rho = 1$.

Using the above expressions for $\mathbf{E}_v(\boldsymbol{r})$ and $\mathbf{E}_h$, the function $\mathbf{E}_{vh}(\boldsymbol{r})$ becomes

$$\mathbf{E}_{vh}(\boldsymbol{r}, 0; \boldsymbol{L}_1, \boldsymbol{L}_2) = \log(N) + 2(\tilde{E} - \tilde{E}_0) + f_e(\boldsymbol{r}) - f_L(\boldsymbol{r}) \tag{12}$$

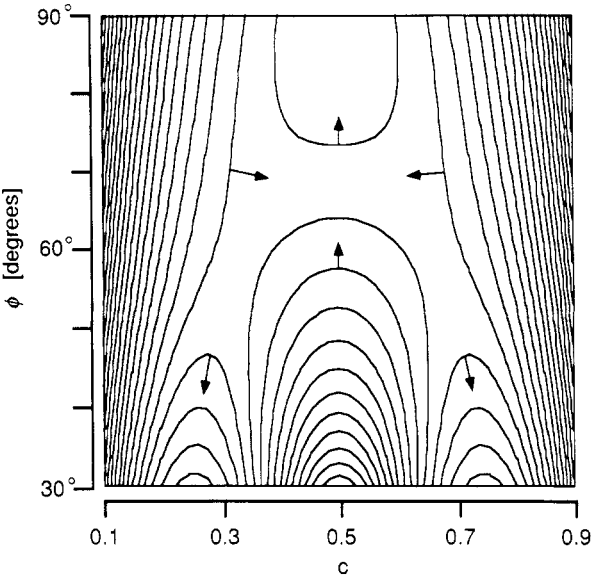


**Figure 3.** $\mathbf{E}_v(\boldsymbol{r};\boldsymbol{e}_1,\boldsymbol{e}_2)$ evaluated on the diagonal $\boldsymbol{r}=c(\boldsymbol{e}_1+\boldsymbol{e}_2)$ of unit cells with $|\boldsymbol{e}_1|/|\boldsymbol{e}_2|=1$ and $30°\leq\phi\leq 90°$. The arrows point in the direction of decreasing energy. Two minima appear on the diagonal of the unit cell for all $\phi$ less than about 70°.

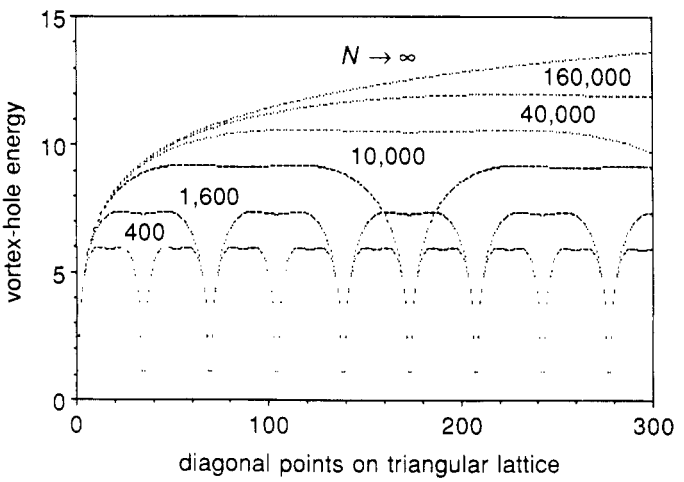


**Figure 4.** $\mathbf{E}_{vh}(\boldsymbol{r};\boldsymbol{L}_1,\boldsymbol{L}_2)$ evaluated on various unit cells $[\boldsymbol{L}_{1,2}=(N)^{1/2}\boldsymbol{e}_{1,2}]$ of the triangular lattice at discrete points $\boldsymbol{r}_{1/3}=(c+n)(\boldsymbol{e}_1+\boldsymbol{e}_2)$, where $c=\frac{1}{3},\frac{2}{3}$ and $0\leq n\leq(N)^{1/2}$. That is, the points lie on the long diagonal of the unit cell at the minima of the elementary unit cells. The units of the $x$ axis are $|\boldsymbol{e}_1|$. The two maxima between the vacancies come from $f_L(\boldsymbol{r})$ in (12), as evident from figure 3.

where $\boldsymbol{r}_1$ has been chosen as the origin, without loss of generality. $\mathbf{E}_{vh}$ is shown in figure 4 for variously sized unit cells with $n_1 = n_2$ on the triangular lattice. The values of $\boldsymbol{r}$ are confined to the discrete values $\boldsymbol{r}_{1/3}$ on the diagonals of the unit cells.

## 4. Applications

### *4.1. An isolated, displaced vortex and its normal modes*

Thus far, the displaced vortices have themselves been superlattices with generators $\boldsymbol{L}_1$, $\boldsymbol{L}_2$. Consider now the case of an *isolated* displaced vortex when $\boldsymbol{L}_1, \boldsymbol{L}_2 \to \infty$,

$$\mathbf{E}^{\infty}_{vh}(\boldsymbol{r}; \boldsymbol{e}_1, \boldsymbol{e}_2) \equiv \lim_{L_1, L_2 \to \infty} \mathbf{E}_{vh}(\boldsymbol{r}; \boldsymbol{L}_1, \boldsymbol{L}_2) = \lim_{n_1, n_2 \to \infty} \mathbf{E}_{vh}(\boldsymbol{r}; n_1\boldsymbol{e}_1, n_2\boldsymbol{e}_2). \tag{13}$$

This limit, which only affects $\log(N) - f_L(\boldsymbol{r})$ in (12), is the same for any $n_1/n_2$:

$$\begin{aligned}\mathbf{E}^{\infty}_{vh}(\boldsymbol{r}) &= f_e(\boldsymbol{r}) + \log[4\pi^2(x^2+y^2)/e_1^2] - \frac{\sigma_0}{6} + 2\log\prod_{k=1}^{\infty}[1 - 2\cos(k\chi_0)\,e^{-k\sigma_0} + e^{-2k\sigma_0}] \\ &= f_e(\boldsymbol{r}) + \log[(x^2+y^2)/e_1e_2\sin\phi] - 2\tilde{E}_0 \end{aligned} \tag{14}$$

where it was convenient to define

$$\sigma \equiv 2\pi \sin\phi/\rho \qquad \chi \equiv 2\pi\cos\phi/\rho. \tag{15}$$

$\mathbf{E}^{\infty}_{vh}(\boldsymbol{r})$ is easy to interpret as the energy of the displaced vortex with the lattice, plus its interaction with a vortex of opposite strength at the vacancy, plus a constant that ensures $\lim_{r\to 0} \mathbf{E}^{\infty}_{vh}(\boldsymbol{r}) = 0$. The envelop function in figure 4 is $\mathbf{E}^{\infty}_{vh}(\boldsymbol{r})$ evaluated at the same points as $\mathbf{E}_{vh}(\boldsymbol{r})$. A similar expression for $\mathbf{E}^{\infty}_{vh}(\boldsymbol{r})$ was obtained by O'Neil [1].

At large vortex-hole separations the overall energy increases as $\log(r)$, with singular maxima and local minima arising from the vortex-lattice interaction $f_e(\boldsymbol{r})$. However, in the vicinity of the vacancy many local minima have been lost, as shown in figure 5 for square and triangular lattices. The minima were found numerically using the conjugate gradient method. In figure 6 the constant energy contours are also shown, which represent dynamical trajectories of the displaced vortex.

It is possible now to examine $\mathbf{E}^{\infty}_{vh}(\boldsymbol{r})$ for small displacements by expanding (14) at $\boldsymbol{r} = 0$ to order $r^2$:

$$\mathbf{E}^{\infty}_{vh}(\boldsymbol{r}) = \left(\frac{2\pi}{e_1}\right)^2\left(\frac{y^2}{\sigma_0} + (x^2-y^2)[\tfrac{1}{12} - 2S_2(\sigma_0, \chi_0)] + xy\,4S_1(\sigma_0, \chi_0)\right) + O(r^3) \tag{16}$$

where $S_{1,2}$ are rapidly converging sums

$$\begin{aligned} S_1(\sigma, \chi) &= \sum_{k=1}^{\infty} \frac{\sin(k\chi)\,e^{-k\sigma}(1 - e^{-2k\sigma})}{[1 - 2\cos(k\chi)\,e^{-k\sigma} + e^{-2k\sigma}]^2} \\ S_2(\sigma, \chi) &= \sum_{k=1}^{\infty} \frac{e^{-k\sigma}[\cos(k\chi)(1 + e^{-2k\sigma}) - 2\,e^{-k\sigma}]}{[1 - 2\cos(k\chi)\,e^{-k\sigma} + e^{-2k\sigma}]^2}. \end{aligned} \tag{17}$$

Unless the lattice is very distorted, the point $\boldsymbol{r} = 0$ is a minimum of the energy and the equipotential surfaces are ellipses. For both the cases of the triangular and the square lattice ($\phi = \pi/3$ and $\pi/2$, and $\chi_0 = \pi$ and 0, respectively) the term $\sin(k\chi_0)$ and, consequently, the sum $S_1$ in (16) vanish, which, in turn, imply that the principal axes of both ellipses coincide with the $xy$ axes. Because the respective six- and four-fold

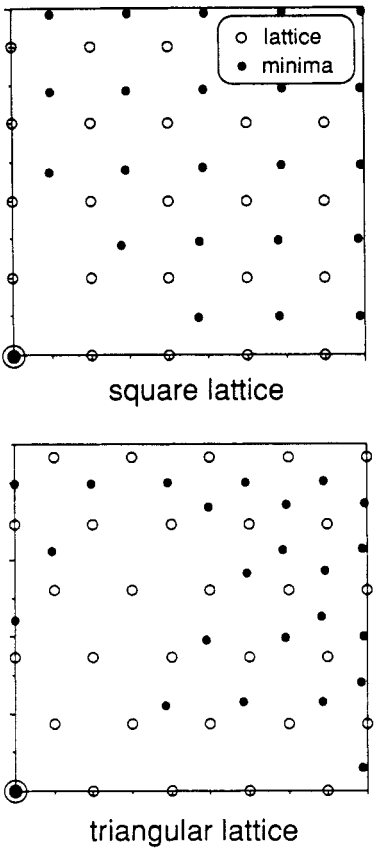


**Figure 5.** Local minima (full circles) of the vortex-hole energy in the vicinity of an isolated vacancy (lower left corner) in a square (top) and triangular (bottom) lattice. Note the absence of minima near the vacancy.

symmetries of these lattices are higher than the symmetry of an ellipse, the ellipses must be circles. Therefore, the RHS of (16) has the form $c(x^2+y^2)$, which implies $S_2=\frac{1}{4}(\frac{1}{6}-1/\sigma_0)$ for triangular and square lattices, giving

$$\mathbf{E}^{\infty}_{\mathrm{vh}}(\boldsymbol{r})=\frac{\pi}{e_1^2}\times\begin{cases}(2/\sqrt{3})(x^2+y^2)+\mathrm{O}(r^3) & \text{triangular lattice}\\ (x^2+y^2)+\mathrm{O}(r^3) & \text{square lattice.}\end{cases} \tag{18}$$

More generally, in figure 7 the small displacement eigenvalues from the numerical solution of (16) are shown as a function of $\phi$ for $\rho=1$. The values are divided by the eigenvalues of the triangular lattice. One of the eigenvalues becomes negative for $\phi\simeq 29.34°$. One may conclude that the single-vortex displacement enjoys first-order stability in most simple lattices.

A heuristic interpretation can be given to the normal modes in (18) by supposing that the displaced vortex interacts with the background in a particular way. Namely, the background is taken to be a disk (of arbitrarily large radius) centred at the displaced vortex's equilibrium position. When the vortex is displaced by $\delta r$ the only restoring force of order $\delta r$ comes from interaction with the portion of the background within a disk of radius $\delta r$. (As seen in section 1, the other vortices only contribute at higher

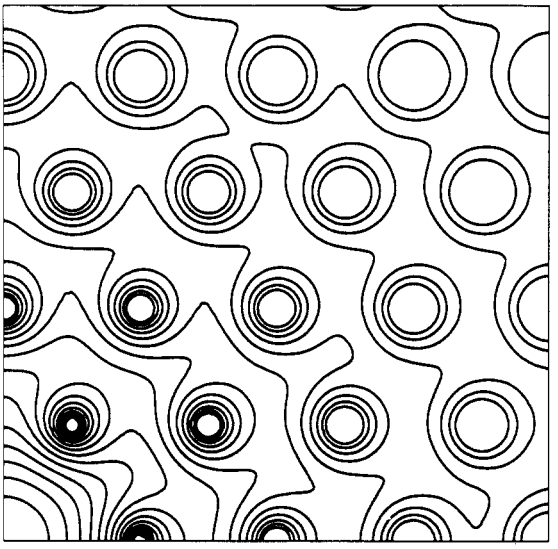

**Figure 6.** Equal-energy contours of the vortex-hole energy near an isolated vacancy in the triangular lattice.

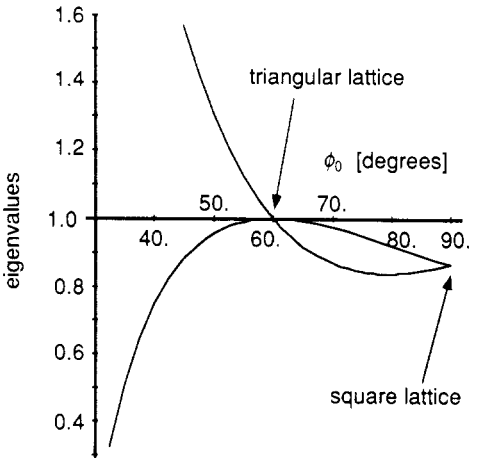


**Figure 7.** Eigenvalues of a single vortex for small displacements from its lattice site as a function of $\phi_0$, for $|e_1|/|e_2| = 1$. The values are normalised to those of the triangular lattice in (18).

order.) By symmetry, the interaction with the background outside the disk of radius $\delta r$ vanishes. Explicitly, the restoring force $\boldsymbol{F}$ is the integral over the background area $A_B$

$$\boldsymbol{F}(\boldsymbol{r}) = 2\rho\Gamma_B\Gamma \iint \mathrm{d}A_B \frac{\boldsymbol{r} - \boldsymbol{r}_B}{|\boldsymbol{r} - \boldsymbol{r}_B|^2}$$

where $\rho$ is the density of the background vorticity and $\Gamma_B = \Gamma = -1$. This gives $\boldsymbol{F} = -2\rho\pi\boldsymbol{r}/r^2$ and a displacement energy $E = \rho\pi r^2$. The results of (18) are thereby

reproduced, because $\rho = 2/\sqrt{3}$ (respectively, 1) for a triangular (respectively, square) lattice.

To answer the 'puzzle' posed in section 1, it is indeed the background that gives stability. The lattice energy equation is derived as the joint limit of finite vortex systems plus the background. For the finite systems stability is manifest, although for no finite system is the triangular lattice the arrangement of lowest energy. If the lattice were neutral the background would not be present, but systems of mixed $\Gamma$ (i.e. both + and −) are never stable, whether or not they are neutral.

### *4.2. An isolated, displaced vortex row and its normal modes*

A more general expression for small displacements is obtained by expanding prior to the $N \to \infty$ limit, i.e. by expanding (12):

$$\mathbf{E}_{vh}(\boldsymbol{r}) = \left(\frac{2\pi}{e_1}\right)^2 \left( \frac{y^2}{\sigma_0}(1 - 1/n_1 n_2) + (x^2 - y^2)[\tfrac{1}{12}(1 - 1/n_1^2) - 2(S_2(\sigma_0, \chi_0) - S_2(\sigma, \chi)/n_1^2)] \right.$$
$$\left. + xy4(S_1(\sigma_0, \chi_0) - S_1(\sigma, \chi)/n_1^2) \right) + O(r^3). \tag{19}$$

If $n_1 = n_2$ then the above can be written as $\mathbf{E}_{vh}(\boldsymbol{r}) = (1 - 1/N)\mathbf{E}_{vh}^{\infty}(\boldsymbol{r}) + O(r^3)$. The vortex-hole 'binding energy' decreases as the unit cell decreases until it identically vanishes at $N = 1$, which corresponds to uniform displacement of the lattice.

The energy required to displace every $n_1$th vortex on a single row parallel to the $x$ axis corresponds to the limit $n_2 \to \infty$ in (19). In particular, the energy for the solid row, $n_1 = 1$, is

$$\mathbf{E}_{vh}^{row}(\boldsymbol{r}) = \left(\frac{2\pi}{e_1}\right)^2 \left( \frac{y^2}{\sigma_0} - 2(x^2 - y^2)S_2(\sigma_0, \chi_0) + xy4S_1(\sigma_0, \chi_0) \right) + O(r^3). \tag{20}$$

For the triangular and square lattices this infinite row displacement energy becomes

$$\mathbf{E}_{vh}^{row}(\boldsymbol{r}) = \frac{\pi}{e_1^2} \times \begin{cases} x^2\left(\frac{2}{\sqrt{3}} - \frac{\pi}{3}\right) + y^2\left(\frac{2}{\sqrt{3}} + \frac{\pi}{3}\right) + O(r^3) & \text{triangular lattice} \\ x^2\left(1 - \frac{\pi}{3}\right) + y^2\left(1 + \frac{\pi}{3}\right) + O(r^3) & \text{square lattice.} \end{cases} \tag{21}$$

Note that for the square lattice the row is unstable to displacement in the $x$ direction, but for either lattice the row is more stable to displacements in the $y$ direction than is a single vortex, as given by (18).

### *4.3. An isolated, displaced vortex cluster*

Consider now a cluster of $M$ vortex-hole pairs. From the same expansion used in (14) it is seen that in the limit $n_1, n_2 \to \infty$ the energy given in (5) for several displaced vortices becomes

$$\mathbf{E}_{vh}^{\infty}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_M, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_M)$$
$$= \lim_{n_1, n_2 \to \infty} \boldsymbol{E}_{vh}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_M, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_M)$$
$$= -M[2\tilde{E}_0 + \log(e_1 e_2 \sin\phi)] + \sum_{j=1}^{M} f_e(\boldsymbol{w}_j)$$
$$+ \sum_{i=1}^{M}\sum_{j=1}^{M} \log|\boldsymbol{w}_i - \boldsymbol{r}_j|^2 - \sum_{i<j}^{\lfloor M \rfloor}\sum \log(|\boldsymbol{w}_i - \boldsymbol{w}_j|^2|\boldsymbol{r}_i - \boldsymbol{r}_j|^2) \tag{22}$$

where the periodicity of $f_e(\boldsymbol{w}_i - \boldsymbol{r}_j)$ was used and the origin was assumed to be a lattice point. Therefore, $M$ displaced vortices in an isolated cluster interact with the lattice according to $\mathbf{E}_v(\boldsymbol{w}_j)$ and among themselves as $M$ free vortices (with the same circulation) in the presence of the same number of vortices at fixed lattices sites $\boldsymbol{r}_j$ (with opposite circulation). The dynamics of these displaced vortices, which is largely to be explored, is given by the usual formulae

$$\begin{bmatrix} \dot{x}_j \\ \dot{y}_j \end{bmatrix} = \begin{bmatrix} \partial/\partial y_j \\ -\partial/\partial x_j \end{bmatrix} \mathbf{E}^{x}_{vh}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_M, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \qquad \boldsymbol{w}_j = (x_j, y_j). \tag{23}$$

### *4.4. Superlattice and isolated vortex clusters displaced on a general lattice*

Finally, consider the unrestricted case of $J$ vortices per elementary unit cell with arbitrary strengths $\Gamma_j$. Denote the vortex positions in a general superlattice unit cell having generators $\boldsymbol{L}_1 = n_1\boldsymbol{e}_1$, $\boldsymbol{L}_2 = n_2\boldsymbol{e}_2$ by $\boldsymbol{r}_{kj}$, where $k$ denotes the elementary cell and $j$ denotes the species: $1 \leq k \leq n_1 n_2 \equiv K$ and $1 \leq j \leq J$. The energy of a superlattice of displacements $\boldsymbol{r}_{11} \to \boldsymbol{r}$ is

$$\begin{aligned} &\mathbf{E}_{vh}(\boldsymbol{r}, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) \\ &\quad \equiv NE(\boldsymbol{r}, \boldsymbol{r}_{12}, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) - NE(\boldsymbol{r}_{11}, \boldsymbol{r}_{12}, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2) \\ &\quad = \sum_{k=1}^{K} \sum_{j=1}^{J} \Gamma_1\Gamma_j f_L(\boldsymbol{r} - \boldsymbol{r}_{kj}) - \sum_{k=1}^{K} \sum_{j=1}^{J}{}' \Gamma_1\Gamma_j f_L(\boldsymbol{r}_{11} - \boldsymbol{r}_{kj}) - \Gamma_1^2 f_L(\boldsymbol{r} - \boldsymbol{r}_{11}) \end{aligned} \tag{24}$$

where $N = KJ$ and the prime on the second summation omits the term $\boldsymbol{r}_{kj} = \boldsymbol{r}_{11}$. The two summations over $f_L$ can be transformed to expressions in $f_e$ by the identity in (10) and

$$f_e(\boldsymbol{r}_{1i} - \boldsymbol{r}_{1j}) = \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{K} f_L(\boldsymbol{r}_{ki} - \boldsymbol{r}_{lj}) = \sum_{l=1}^{K} f_L(\boldsymbol{r}_{li} - \boldsymbol{r}_{lj}) \qquad i \neq j \tag{25}$$

which is the equivalent of (10). The generalisation of (12) then becomes

$$\begin{aligned} &\mathbf{E}_{vh}(\boldsymbol{r}, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2 | \Gamma_1) \\ &\quad = \Gamma_1^2[\log(K) + 2(\tilde{E} - \tilde{E}_0) + f_e(\boldsymbol{r} - \boldsymbol{r}_1) - f_L(\boldsymbol{r} - \boldsymbol{r}_1)] \\ &\qquad + \sum_{j=2}^{J} \Gamma_1\Gamma_j[f_e(\boldsymbol{r} - \boldsymbol{r}_j) - f_e(\boldsymbol{r}_1 - \boldsymbol{r}_j)]. \end{aligned} \tag{26}$$

The limit $n_1, n_2 \to \infty$ affects only $f_L$ and $\log(K)$, as in (14)

$$\begin{aligned} \mathbf{E}^{x}_{vh}(\boldsymbol{r}|\Gamma_1) &= \Gamma_1^2\{f_e(\boldsymbol{r}) + \log[(x^2 + y^2)/e_1 e_2 \sin\phi] - 2\tilde{E}_0\} \\ &\quad + \sum_{j=2}^{J} \Gamma_1\Gamma_j[f_e(\boldsymbol{r} - \boldsymbol{r}_j) - f_e(\boldsymbol{r}_1 - \boldsymbol{r}_j)]. \end{aligned} \tag{27}$$

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## Appendix

Here we give the explicit expression for the energy per vortex of an infinite vortex lattice having unit cell generators $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$. The interaction energy between two vortices is taken to be $-\Gamma_i\Gamma_j \log|\boldsymbol{r}_i - \boldsymbol{r}_j|^2$. (No intrinsic or 'core' energy per vortex is included because it is both non-generic and trivial to add (so long as the cores do not overlap). As a result, the energy is defined only to an additive constant.)

A unit cell contains $N$ vortices of arbitrary strength $\Gamma_j$ at arbitrary positions $\boldsymbol{r}_j$. The generator $\boldsymbol{L}_2$ is at an angle $\phi$ with respect to $\boldsymbol{L}_1$, which is parallel to the $x$ axis. As stated earlier, it is convenient to define

$$\sigma \equiv 2\pi \sin\phi/\rho \qquad \chi \equiv 2\pi \cos\phi/\rho \tag{A1}$$

where

$$\rho = L_1/L_2 \qquad \boldsymbol{L}_1 \cdot \boldsymbol{L}_2 = L_1 L_2 \cos\phi.$$

The useful relation between $x, y$ and unit cell coordinates $q_1, q_2$ is

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} = q_1\boldsymbol{L}_1 + q_2\boldsymbol{L}_2 = (x - y\chi/\sigma)\boldsymbol{L}_1/|\boldsymbol{L}_1| + (2\pi y/\sigma)\boldsymbol{L}_2/|\boldsymbol{L}_2|. \tag{A2}$$

The energy per vortex is [2-4]

$$E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{L}_1, \boldsymbol{L}_2)$$

$$= \overline{\Gamma^2}\tilde{E}(\phi, \rho) + \tfrac{1}{2}\overline{\Gamma^2}\log(N) + \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Gamma_i\Gamma_j f(\boldsymbol{r}_i - \boldsymbol{r}_j; \boldsymbol{L}_1, \boldsymbol{L}_2) \tag{A3}$$

where

$$\tilde{E}(\phi, \rho) = \tfrac{1}{12}\sigma - \tfrac{1}{2}\log(2\pi\sigma) - \log\prod_{p=1}^{\infty} g_p(0, \sigma, \chi) \tag{A4}$$

$$f(\boldsymbol{r}; \boldsymbol{L}_1, \boldsymbol{L}_2) = \frac{2\pi|y|}{L_1}\left(\frac{2\pi|y|}{L_1\sigma} - 1\right) + \frac{\sigma}{6} - \log\prod_{p=-\infty}^{\infty} g_p(\boldsymbol{r}; \sigma, \chi)$$

$$= \sigma[|q_2|(|q_2| - 1) + \tfrac{1}{6}] - \log\prod_{p=-\infty}^{\infty} g_p(\boldsymbol{r}; \sigma, \chi) \tag{A5}$$

and

$$\overline{\Gamma^2} = \frac{1}{N}\sum_{j=1}^{N}\Gamma_j^2$$

$$g_p(\boldsymbol{r}; \sigma, \chi) = 1 - 2\cos(2\pi x/L_1 + p\chi)\exp(-|\sigma p + 2\pi y/L_1|) + \exp(-2|\sigma p + 2\pi y/L_1|)$$

$$= 1 - 2\cos[2\pi q_1 + (p + q_2)\chi]\exp(-\sigma|p + q_2|) + \exp(-2\sigma|p + q_2|). \tag{A6}$$

In practice, only three to five terms of the infinite products are needed for high accuracy. The expression for $f(\boldsymbol{r}; \boldsymbol{L}_1, \boldsymbol{L}_2)$ in (A5) is not explicitly periodic in the $\boldsymbol{L}_2$ direction because a relation, exact only for $0 \leq y < |\boldsymbol{L}_2|\sin\phi$, was used to simplify the fully periodic expression [4]. Therefore, $\boldsymbol{r}$ should be taken modulo $\boldsymbol{L}_2$ when using (A5).

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